ANSWERS TO PROF. JOHNSON'S QUERY IN NO. 3.—"QUERY. Let $u = \frac{\sin ax}{a}$. Now if $a = \infty$, $u = 0$ independently of the value of x, therefore we should have $\frac{du}{dx} = 0$ when $a = \infty$. But we find $\frac{du}{dx} =$ cosine ax which is essentially indeterminate when $a = \infty$. What is the explanation of this paradox?"

BY R. J. ADCOCK.—When $u = 0$ independently of x it is not a function of x, and therefore cannot be differentiated with respect to x. Therefore the value of $du \div dx$ is $\cos ax$ independently of the value of a.

BY PROF. JUDSON.—If $u = 0$, independently of x, then u is not a function of x, and $du \div dx$ is without meaning.

If a is a constant, then a cannot $= \infty$. If a is a variable, independent of x, and $a = \infty$, i. e. a increases without limit, then $\frac{\sin ax}{a} =$ an infinitesimal (not $= 0$), and u is therefore indeterminate; $du \div dx = \cos ax$ is also indeterminate, and there is no paradox.

BY PROF. BARBOUR. Let $u = \frac{\sin 360x}{360}$; $\frac{du}{dx} = \cos 360x$. Now for $x = 1°$, or $2°$ or any other integral number of degrees, $u = 0$; $\frac{du}{dx} = \cos 0° = 1$. Hence it is clear that u may be equal to 0, and yet $du \div dx = 1$.

SOLUTIONS OF PROBLEMS.

---

342. *By Prof. Kershner.*—"Prove Schlömilch's Theorem: If D_a, D_b, .. D_n are divisors of 10^k+1, so that $N_a = \frac{10^k+1}{D_a}$, $N_b = \frac{10^k+1}{D_b}$, $N_n = \frac{10^k+1}{D_k}$ the k digits or figures of the whole numbers $D_a - 1$, $D_b - 1$, $D_n - 1$ are the k first figures of the circulator or period of $\frac{1}{N_a}$, $\frac{1}{N_b}$, $\frac{1}{N_n}$, respectively.

SOLUTION BY PROF. J. SCHEFFER, HARRISBURGH, PA.

A well known principle relating to circulating decimals is as follows:

If $1 \div M$ *produces a period of* 2k *dicimals, the remainder after the* kth *digit is* $(M - 1)$, *and the following decimals can be obtained by subtracting each digit from* 9.

Now let $10^k + 1 = MN$, where M and N may be prime factors or the product of such; and denote by x the number which represents the first k

decimals, then we have obviously according to the above principle:

$$\frac{1}{M}=\frac{x}{10^k}+\frac{M-1}{10^k M},\ \text{whence}\ x=\frac{(10^k+1)-M}{M},$$

but $10^k+1=MN$, $\therefore x=(MN-M)\div M=N-1$, which proves the theorem.

SOLUTIONS of problems in No. 3 have been received as follows:

From R. J. Adcock, 352; Marcus Baker, 351; Prof. W. P. Casey, 346, 349, 350, 351; Prof. H. T. Eddy, 354; Prof. W. W. Hendrickson, 346; O. L. Mathiot, 351; Prof. E. B, Seitz, 349, 351, 352, 353; Prof. J. Scheffer, 347, 348, 351; R. S. Woodward, 347, 348.

---

346. "Chords of the parabola $y^2=4ax$ are drawn through the fixed point (h, k); required the locus of the intersection of normals drawn at the extremities of the chord."

SOLUTION BY PROF. W. W. HENDRICKSON.

Let the equation to the chord be $y_1-k=m_1(x_1-h)$ .... (1), and the equation to the parabola $y_1^2=4ax$ .. (2); combining (1) and (2) we have

$$y_1^2m_1-4ay_1+4ak-4am_1h=0.$$

Let the roots of this equation be α and β, then $\alpha+\beta=\frac{4a}{m_1}$, $\alpha\beta=\frac{4a(k-m_1h)}{m_1}$

Let m_2, m_3 be the direction ratios of the normals, then

$$m_2=\frac{-\alpha}{2a},\ m_3=\frac{-\beta}{2a},\ m_2m_3=\frac{k-m_1h}{am_1},\ m_2+m_3=\frac{-2}{m_1}$$

Taking the origin at $(2a, 0)$, the equation to the normal is $y=mx-am^3$; Denote the roots of this equation by m_2, m_3, m_4; then $m_2+m_3+m_4=0$, but $m_2+m_3=\frac{-2}{m_1}$, hence $m_4=\frac{2}{m_1}$, also $m_2m_3m_4=\frac{-y}{a}=\frac{k-m_1h}{am_1}\,\frac{2}{m_1}$, or $y=\frac{2(m_1h-k)}{m_1^2}$. Substituting this value of y in the equation to the normal we find $x=h+\frac{4a-km_1}{m_1^2}$; or moving the origin again to $(h, 0)$ we have

$$y=\frac{2(m_1h-k)}{m_1^2},\qquad x=\frac{4a-km_1}{m_1^2}.$$

Finally eliminating m_1 between these two equations, the equation to the required locus is

$$(4ah-k^2)(ky+2hx)=2(2ay+kx)^2.$$

---

347. "Given $z=a\sin(x+\alpha)+b\sin(y+\beta)$, reduce z to the form

$$z=D\sin\tfrac{1}{2}(x+\alpha+y+\beta+\delta)."$$

SOLUTION BY R. S. WOODWARD, U. S. LAKE SURVEY, DETROIT, MICH.

Put $a = c+d$ and $b = c-d$. Then

$$z = c[\sin(x+\alpha)+\sin(y+\beta)]+d[\sin(x+\alpha)-\sin(y+\beta)]$$
$$= 2c\sin\tfrac{1}{2}(x+\alpha+y+\beta)\cos\tfrac{1}{2}(x+\alpha-y-\beta)$$
$$+\ 2d\cos\tfrac{1}{2}(x+\alpha+y+\beta)\sin\tfrac{1}{2}(x+\alpha+y+\beta).$$

Now put $2c\cos\frac{1}{2}(x+\alpha-y-\beta) = D\cos\frac{1}{2}\delta$,

$2d\sin\frac{1}{2}(x+\alpha-y-\beta) = D\sin\frac{1}{2}\delta$, and we get

$$z = D\sin\tfrac{1}{2}(x+\alpha+y+\beta+\delta).$$

---

348. "Show how to determine the values of x and z which will render

$$\begin{aligned} u = &+2\alpha_1\cos(\ qz+\tfrac{1}{2}qx+\beta_1)\sin\tfrac{1}{2}qx \\ &+2\alpha_2\cos(2qz+\ qx+\beta_2)\sin\ qx \\ &+2\alpha_3\cos(3qz+\tfrac{3}{2}qx+\beta_3)\sin\tfrac{3}{2}qx \\ &+\ \ldots\ldots\ldots\ldots \\ &+2\alpha_n\cos(nqz+\tfrac{n}{2}qx+\beta_n)\sin\tfrac{n}{2}qx, \end{aligned}$$

a max. or min., α_1, α_2, etc., β_1, β_2, etc. and q being constants."

SOLUTION BY R. S. WOODWARD.

For brevity this expression may be written

$$u = 2\Sigma_{r=1}^{r=n}\alpha_r\cos(rqz+\tfrac{1}{2}rqx+\beta_r)\sin\tfrac{1}{2}rqx.$$

Hence for a max. or min.

$$(1)\qquad \frac{du}{dx} = 2q\sum_{r=1}^{r=n}\left\{\begin{array}{l} r\alpha_r\cos(rqz+\frac{1}{2}rqx+\beta_r)\cos\frac{1}{2}rqx\ - \\ r\alpha_r\sin(rqz+\frac{1}{2}rqx+\beta_r)\sin\frac{1}{2}rqx \end{array}\right\}$$
$$= 2q\Sigma_{r=1}^{r=n}r\alpha_r\cos(rqz+\ rqx+\beta_r) = 0.$$

$$(2)\qquad \frac{du}{dz} = -2q\sum_{r=1}^{r=n}r\alpha_r\sin(rqz+\tfrac{1}{2}rqx+\beta_r)\sin\tfrac{1}{2}rqx = 0.$$

Subtracting twice (2) from (1) there results

$$(3)\qquad 2q\Sigma_{r=1}^{r=n}r\alpha_r\cos(rqz+\beta_r) = 0.$$

The last eq'n will give the critical values of z. Denote them by z_1, z_2, $\ldots z_m$. Then the critical values of x corresponding to z_1 will be 0, (z_2-z_1), (z_3-z_1), $\ldots (z_m-z_1)$, since z_1 and either of these values of x will satisfy (1). Collectively the critical values of x corresponding to any critical value of z are shown in the following table:

Crit. val's of z	Corresponding critical values of x			
z_1	0,	(z_2-z_1),	(z_3-z_1),	$\ldots (z_m-z_1)$
z_2	(z_1-z_2),	0,	(z_3-z_2),	$\ldots (z_m-z_2)$
z_3	(z_1-z_3),	(z_2-z_3),	0,	$\ldots (z_m-z_3)$
. .	.	.	. . . .	. . .
z_m	(z_1-z_m),	(z_2-z_m),	. . . . .	0

It may be remarked that the series u, expresses in general terms the correction for periodic error to the observed value of an angle measured on a circle read by q equidistant microscopes, z being the reading of either microscope and x the angle observed, or the difference between the means of the microscope readings in their two positions. When for any instrument the values of the constants α_1, α_2, etc., β_1, β_2, etc., are known, it may be important to know what values of z and x will render u a max. or min.

The practical application of equation (3) presents no special difficulty, since the roots z_1, z_2, etc., are not generally required with any great precision. By computing for each term of (3) its value for a few equidistant intervals throughout its period, the curve represented by

$$\Sigma_{r=1}^{r=n} r\alpha_r \cos(rqz + \beta_r) = y, \text{ say,}$$

may be plotted and the values of z, making $y = 0$, readily detected.

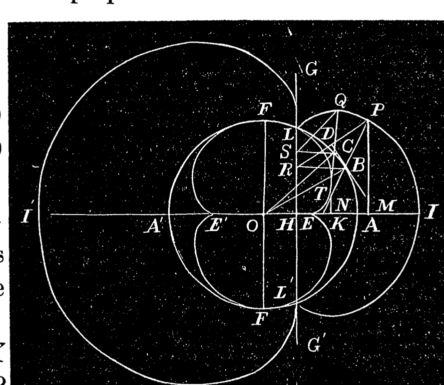
---

349. "From any point B of a circle, whose radius is a, a perpendicular BR is drawn to a fixed straight line whose distance from the centre is b; and from R a perpendicular RD is drawn to the tangent at B. Produce RD to P making $DP = RD$. Find the rectangular equation of the locus of P, and of the evolute of this locus."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $AFA'F'$ be the given circle, and GG' the given straight line. Draw the diameter AA' perpendicular to GG', and FF' parallel to GG'.

Let B, C be two consecutive points in the circumference, and P, Q the corresponding points in the locus, CS being drawn perpendicular to GG', and SQ perpendicular to the tangent at C. Produce PB to K in OA, and QC to T in BK, and draw PM and TN perpendicular to OA.



Let $OA = a$, $OH = b$, $OM = x$, $PM = y$, $ON = x'$, $TN = y'$, $\angle AOB = \theta$, and arc $BC = i$.

Then since $\angle CBR = 90° - \theta = BCT$, $CQ = CS = BR - i \sin\theta = BP - i\sin\theta$, and $CT = BT + i\sin\theta$; hence $CQ + CT = BP + BT$, or $QT = PT$. Therefore T is the point in the evolute of the curve corresponding to P.

We have $BP = a\cos\theta - b$, $BK = OK = \frac{1}{2}a\sec\theta$, $MK \cdot \sin MKP$

$= PM \sin MPK$, or $(x-\frac{1}{2}a \sec\theta)\sin 2\theta = y\cos 2\theta$, whence

$$x\sin 2\theta - y\cos 2\theta = a\sin\theta. \tag{1}$$

We also have $MK\cos MKP + PM\cos MPK = PK$, or

$$(x-\tfrac{1}{2}a\sec\theta)\cos 2\theta + y\sin 2\theta = \tfrac{1}{2}a\sec\theta + a\cos\theta - b, \text{ whence}$$

$$x\cos 2\theta + y\sin 2\theta = 2a\cos\theta - b. \tag{2}$$

The sum of the squares of (1) and (2) gives

$$3a^2\cos^2\theta - 4ab\cos\theta - (x^2+y^2-a^2-b^2) = 0.$$

The sum of (1) multiplied by $\sin 2\theta$, and (2) multiplied by $\cos 2\theta$ gives

$$2a\cos^3\theta - 2b\cos^2\theta - (x-b) = 0. \tag{4}$$

Subtracting (3) multiplied by $2\cos\theta$, from (4) multiplied by $3a$ we have

$$2ab\cos^2\theta + 2(x^2+y^2-a^2-b^2)\cos\theta - 3a(x-b) = 0. \tag{5}$$

Subtracting (3) multiplied by $2b$, from (5) multiplied by $3a$, we have

$$[6(x^2+y^2-a^2-b^2)+8b^2]\,a\cos\theta - [9a^2(x-b) - 2b(x^2+y^2-a^2-b^2)] = 0. \tag{6}$$

Subtracting (5) multiplied by $x^2+y^2-a^2-b^2$, from (3) multiplied by $3a(x-b)$, we have

$$[9a^2(x-b) - 2b(x^2+y^2-a^2-b^2)]a\cos\theta - [12a^2b(x-b) + 2(x^2+y^2-a^2-b^2)^2] = 0. \tag{7}$$

From (6) and (7) we find

$$4(x^2+y^2-a^2)(x^2+y^2-a^2-b^2)^2 + 36a^2b(x-b)(x^2+y^2-a^2-b^2)$$
$$+32a^2b^3(x-b) - 27a^4(x-b)^2 = 0,$$

the equation of the locus of P.

Since the angle $PTQ = 2BOC$, we have $BC : \frac{1}{2}BC\sin BCT :: BO : BT$, or $i : \frac{1}{2}i\cos\theta :: a : BT$, whence $BT = \frac{1}{2}a\cos\theta$. Hence we have

$$KT = \tfrac{1}{2}a\sec\theta - \tfrac{1}{2}a\cos\theta = \tfrac{1}{2}a\sin^2\theta\sec\theta,\ TN = KT\sin TKN, \text{ or}$$

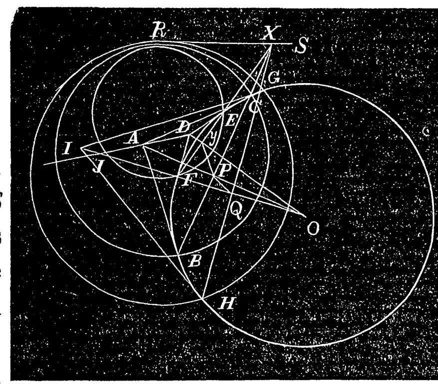
$$y' = \tfrac{1}{2}a\sin^2\theta\sec\theta\sin 2\theta = a\sin^3\theta. \tag{8}$$

We also have $KN = KT\cos TKN = \frac{1}{2}a\sin^2\theta\sec\theta\cos 2\theta$, and

350. "A series of circles touching each other at a point are cut by a fixed circle; show (by third Book of Euclid) that the intersections of the pairs of tangents to the latter, at the points where it is cut by each of the other circles, lie in a straight line."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let RGH, RCB, REP, &c. be the series of circles touching each other at the point R, and O the fixed one intersecting them in the points G, H; C, B; E, F; &c.; I, A, D, &c., the intersections of the tang'ts from these points. Then will I, A, D, &c., be in a straight line.



For if not, allow DA when produced not to pass through I. Join OD, OA and OI intersecting DA in J, and draw the tangent RS. Join FE and produce it to meet RS in x; BC and HG will also pass through x. Join YP and PQ. The figure $OQPYX$ is inscriptible in a circle, the angles OQX, OPX and OYX being right angles, and $\therefore$ $\angle OYP = \angle PQF$; but $OYP = DAP$, because $ADYP$ is also inscriptible in a circle, as $DO \times OY = AO \times OP$, each being equal to R^2. Therefore $\angle PQF = DAP$, whence the figure $APQJ$ is inscriptible in a circle, therefore $AO \times OP = JO \times OQ = R^2 = OI \times OQ$, whence $OI = OJ$ which is impossible; $\therefore$ D, A, I are in the same straight line.

---

351. "In a plane triangle ABC, a line from C perpendicular to AC meets AB in M and another from C perpendicular to BC meets AB in N; knowing the sides a and b and the intercept $MN = m$, it is required to determine the triangle."

SOLUTION BY PROF J. SCHEFFER, HARRISBURGH, PA.

Denoting the angles of the triangle ABC lying opposite the sides a and b by A and B, respectively, and the third side AB, by x, we have $CN = a \tan B$, $CM = b \tan A$. Also $CN \sin B + CM \sin A = m$, therefore

$$a \tan B \sin B + b \tan A \sin A = m, \text{ or}$$

$$\frac{a \sin^2 B}{\cos B} + \frac{b \sin^2 A}{\cos A} = m.$$

Subst'ng for $\sin B$, $\cos B$, &c., their values, found from the sides a, b, x,

$$x^5 + mx^4 - 2(a^2 + b^2)x^3 + (a^2 - b^2)^2 x - m(a^2 - b^2)^2 = 0.$$

352. "Two chords of equal but unknown lengths are drawn at random in a given circle; find the chance of their intersection."

[Prof. Seitz obtains for answer to this problem $2 \div \pi$, while Mr. Adcock gets $\frac{1}{4}\pi$. The difference in these results arises from different conceptions of the problem. In both solutions the first chord is supposed to be drawn from any fixed point to every other point in the semi circumf., and in Mr. Seitz's solution the intersections are supposed to occur at equidistant points on the first chord; while in Mr. Adcock's solution the second chord is supposed to be drawn from equidist. points in the arc subt'd by the first chord.

As these two methods give different results, and no good reason is appaent why one should be adopted rather than the other, we dismiss the quest. for the present and until further discussed by our contributors.

Solutions of 353 and 354 (incorrectly printed 353 at p. 104) will be published in the Sept. No.—Ed.]

PROBLEMS.

355. *By Benj. Headley, Dillsborough, Ind.*—The length of a garden, in the form of a parallelogram, is one rod greater than the breadth. Within the garden is a fountain; and a gravel walk extends diagonally across the garden, from corner to corner, and the distance from the fountain to one end of said walk is three rods, and to the other end four rods; and from this end of the walk, along one end of the garden, to the next corner, and from thence to the fountain, is eight rods. Required the area of the garden.

356. *By Prof. Casey.*—In a triangle ABC, BD is perpendicular to the base AC, and O is the center of gravity of the triangle. Join AO, DO & CO. Given the base AC and the angles AOD, AOC to construct the triangle ABC.

357. *By Prof. De Volson Wood.*—An elastic string without weight and of given length, has one end fixed in a perfectly smooth horizontal plane, and the other to a point in the surface of a sphere, the string being unwound. The sphere is projected on the plane from the fixed point with a linear velocity v and an angular velocity ω, winding the string on the circumference of a great circle; required the elongation of the string when fully stretched, and the subsequent motion of the sphere.

358. *By R. S. Woodward.*—Given the angles A, B and C of a plane triangle, and $d \log a$, $d \log b$ and $d \log c$; a, b, c being the sides respectively.

What are the corresponding values of dA, dB and dC expressed in seconds of arc?